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Valuation of multi-asset option under jump-diffusion process using meshfree moving least squares method

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Abstract

The aim of this article is to develop a meshfree moving least squares collocation method in conjunction with implicit-explicit temporal discretization to numerically solving partial integro-differential equations arising from the valuation of multi-asset options when underlying price processes are modeled by exponential Lévy process.

Keywords: Multi-asset option pricing, Jump-diffusion models, Partial integro-differential equations.
Mathematics Subject Classification [2018]: 45K05, 60G51, 65M06

1 Introduction

The pricing an option written on the underlying asset under the assumption that the dynamic of underlying asset follows the geometric Brownian motion reduces to parabolic PDEs , known as Black-Scholes model. But, empirical studies show that the log return distribution is negatively skewed and has fatter tails than the normal distribution which is assumed for the Black-Scholes model. Recently, the inclusion of jumps, where large returns are represented as jump in prices, into asset price modeling has been developed. Merton [1] has been considered a jump-diffusion model under the assumption that the log–asset price follows a diffusion with jumps that have a normal distribution. As other Lévy processes we can refer to Kou's jump-diffusion model [2] where the jump sizes have a double exponential distribution. These models, which have finite activity, are the interested models in financial markets to capture the phenomenon of the sudden jump which is observed in the real option market. As main contribution of this paper, we apply the meshfree MLS approximation to obtain an accurate valuation multi-asset option pricing where the underlying assets are assumed to have uncorrelated jumps.

2 Moving least squares meshfree method

Consider the predetermined distinct data site $\mathbf{X} = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\} \subset \mathbb{R}^d$ in a neighborhood of $\mathbf{x}$ and data value $\mathbf{f} = \{f(\mathbf{x}_1), \cdots, f(\mathbf{x}_n)\} \subset \mathbb{R}$ evaluated from function u at $\mathbf{X}$. First, we consider the quasi-interplant expansion of the form

$$\mathcal{Q}u(\mathbf{x}) = \sum_{i=1}^{n} u(\mathbf{x}_i)\psi_i(\mathbf{x}), \qquad x \in \mathbb{R}^d. \tag{1}$$

[1] speaker

The goal of MLS approximation is to find the coefficient functions ψ_i, $i = 1, \cdots, n$, in Eq. (1) such that $\mathcal{Q}f$ becomes the best approximation of u at $\mathbf{x}$ in Π_d^m, i.e., the space of m-variate polynomials of total degree at most d with dimension $Q = \begin{pmatrix} d+Q \\ d \end{pmatrix}$. For this goal one can use the Backus-Gilbert formulation of the MLS scheme which attempts to find the function $\psi_i(\mathbf{x})$ at the fixed point $\mathbf{x}$ by minimizing the functional [3]

$$\frac{1}{2}\sum_{i=1}^{n} \psi_i^2(\mathbf{x}) w_i(\mathbf{x}), \tag{2}$$

subject to the polynomial reproduction constraints

$$\sum_{i=1}^{n} p(\mathbf{x}_i)\psi_i(\mathbf{x}) = p(\mathbf{x}), \qquad \forall p \in \Pi_d^m, \tag{3}$$

where w_i, $i = 1, \cdots, n$, are positive radial weight functions whose value increases with the distance from the center. Defining $\mathbf{u} = [u(\mathbf{x}_1), \cdots, u(\mathbf{x}_n)]^T$ the MLS approximation is [4]

$$\mathcal{Q}u(\mathbf{x}) = \sum_{i=1}^{n} u(\mathbf{x}_i)\psi_i(\mathbf{x}) = \Psi^T(\mathbf{x})\mathbf{u}. \tag{4}$$

such that the generator function $\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), \cdots, \psi_n(\mathbf{x})]^T$ obtains as $\Psi(\mathbf{x}) = \mathbf{DP}(\mathbf{P}^T\mathbf{DP})^{-1}\mathbf{p}(\mathbf{x})$, where $\mathbf{p}(\mathbf{x}) = [p_1(x), \cdots, p_Q(x)]^T$ form a basis of Π_d^m, $\phi_i(\mathbf{x}) = 1/w_i(\mathbf{x})$, $\mathbf{D}_{n\times n} = \mathrm{diag}(\phi_1(x), \cdots, \phi_n)$ and $[\mathbf{P}_{n\times Q}]_{ji} = p_j(\mathbf{x}_i)$ where $j = 1, \cdots, Q$ and $i = 1, \cdots, n$. Indeed, the derivative of MLS approximation can be used to approximate the derivative of function u as

$$D^\beta \mathcal{Q}u(\mathbf{x}) = \sum_{i=1}^{n} u(\mathbf{x}_i) D^\beta \psi_i(\mathbf{x}). \tag{5}$$

where β is a multi-index symbol.

3 Main results

For constant $d \in \mathbb{N}$, we consider the dynamics of risky asset $\mathbf{S}(.) = (S_1(.), ..., S_d(.))$ as the following stochastic process $S_i(t) := S_i^0 \exp(rt + X_i(t))$, $\quad i = 1, \cdots, d$. for $i = 1, \cdots, d$ where $\mathbf{S}(.) = (S_1(.), ..., S_d(.))$ denotes initial prices and r is risk free rate and $\mathbf{X}(t) := (X_1(t), ..., X_d(t))$ is d-dimensional Lévy process under risk-natural probability measure $\mathbb{P}$ starting from zero. In the martingale pricing approach the value of European option is defined as discounted conditional expectation of it's terminal pay-off under a risk-neutral probability measure $\mathbb{P}$. With assuming the jump components are independent, the value of an multi-assets European option problem at time t, strike price K, maturity T and pay-off function $V_T(\mathbf{S})$ can be modeled as [5]

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \rho_{ij}\sigma_i\sigma_j S_i S_j \dfrac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{d} S_i \dfrac{\partial V}{\partial S_i} - rV \\ \qquad + \displaystyle\sum_{i=1}^{d}\int_{\mathbb{R}} v_i(d\mathbf{z}_i)[V(t, S_1, \cdots, S_d.(e^{z_i}.\mathbf{e}_i) - V(t, S_1, \cdots, S_d) - (e^{z_i} - 1).S_i\dfrac{\partial V}{\partial S_i}] = 0, \\ V(T, S_1, \cdots, S_d) = V_T(S_1, \cdots, S_d), \end{cases} \tag{6}$$

where σ_i denotes the volatility of S_i, ρ_{ij} is the correlation between the asset's S_i and S_j, $\mathbf{e}_i$ is the unit vector where the ith components is one and zero otherwise, v is a Lévy measure on $\mathbb{R}$ written as $v_i(dz_i) = \lambda_i f(z_i)dz_i$ where λ_i is the intensity of jumps and $f(z_i)$ is the normal distributed jump with mean μ_J and variance σ_J^2. Now by applying the following change of variables [6]

$$\begin{cases} \tau = T - t, , \\ \mathbf{x}_i = \ln(\frac{S_i}{S_0}), \\ u(\tau, \mathbf{x}) = \exp(-r(T-t))V(t, \exp(\mathbf{x})), \\ h(x_1, \cdots, x_d) = V_T(\exp(x_1), \cdots, \exp(x_d)), \end{cases} \tag{7}$$

PIDE (6) can be rewritten as

$$u_\tau = \mathcal{D}(u) + \mathcal{I}(u), \quad (\tau, \mathbf{x}) \in [0, T] \times \mathbb{R}^d, \tag{8}$$

where

$$\begin{aligned} \mathcal{D}(u) &:= \sum_{i,j=1}^{d} \mathbf{k}_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} (r - \frac{1}{2}\sigma_i^2) \frac{\partial u}{\partial x_i}, \\ \mathcal{I}(u) &= \sum_{i=1}^{d} \lambda_i \int_{\mathbb{R}} \Big(u(\tau, \mathbf{x} + y\mathbf{e}_i) - u(\tau, \mathbf{x}) - (\exp(y) - 1) \frac{\partial u}{\partial x_i} \Big) f(y) dy, \end{aligned} \tag{9}$$

where ∇ denotes the gradient symbol and

$$\mathbf{k} = \frac{1}{2} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 & \cdots & \rho\sigma_1\sigma_d \\ \vdots & \ddots & \vdots & \\ \rho\sigma_d\sigma_1 & \rho\sigma_d\sigma_2 & \cdots & \sigma_d^2 \end{bmatrix}. \tag{10}$$

In order to solve Eq. (8) numerically, we have to truncate $\mathbb{R}^d$ to a bounded computational domain $\Omega := [-M, M]^d$ for positive number M.

Using the well-known explicit-implicit temporal approximation to discretize Eq. (8) give us

$$\frac{u(t + \Delta t, \mathbf{x}) - u(t, \mathbf{x})}{\Delta t} = \mathcal{D}(u(t + \frac{\Delta t}{2}, \mathbf{x})) + \mathcal{I}(u(t, \mathbf{x})), \tag{11}$$

where Δt is the time step related to the partition $0 = t_0 < t_1 < t_2 < ... < t_N = T$ of time interval $[0, T]$ and $u(t + \frac{\Delta t}{2}, x) := \frac{1}{2}(u(t, \mathbf{x}) + u(t + \Delta t, \mathbf{x}))$. Rearranging the components at time level t and $t + \Delta t$ yields $\mathbf{H}_+ u(t + \Delta t, \mathbf{x}) = \mathbf{H}_- u(t, \mathbf{x})$ where the operators $\mathbf{H}_+$ and $\mathbf{H}_-$ are defined as

$$\begin{aligned} \mathbf{H}_+ &:= 1 - \frac{\Delta t}{2}\mathcal{D} = 1 - \frac{\Delta t}{2}\Big(\sum_{i,j=1}^{d} \mathbf{k}_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \alpha_i \frac{\partial}{\partial x_i} - \Lambda \Big), \\ \mathbf{H}_- &:= 1 + \frac{\Delta t}{2}\mathcal{D} + \Delta t \mathcal{I} = 1 + \frac{\Delta t}{2}\Big(\sum_{i,j=1}^{d} \mathbf{k}_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \alpha_i \frac{\partial}{\partial x_i} - \Lambda \Big) + \Delta t \mathcal{I}, \end{aligned} \tag{12}$$

Now we use the MLS expansion Eq. (4) for approximate solution of functions $u(\mathbf{x}, t + k\Delta t), k = 0, 1$ as

$$u(\mathbf{x}, t + k\Delta t) \approx \sum_{i=1}^{N} u^k(\mathbf{x}_i) \psi_i(\mathbf{x}), \tag{13}$$

where $N = N_i + N_b$ and $X = X_1 \cup X_2$ is the set of quasi-uniform scattered data where $X_1 = \{x_1, ..., x_{N_i}\} \subset \Omega$ and $X_2 = \{x_{N_i+1}, ..., x_{N_i+N_b}\} \subset \partial\Omega$. Now with the help of collocation approach we obtain the following system of equations yields

$$\sum_{i=1}^{N} u^1(\mathbf{x}_i) \mathbf{H}_+ \psi_i(\mathbf{x}) = \sum_{i=1}^{N} u^0(\mathbf{x}_i) \mathbf{H}_- \psi_i(\mathbf{x}), \tag{14}$$

So starting from initial condition $u^0 = h(\mathbf{x})$, we can solve the linear system of equations Eq. (14) based on previous step. Indeed, the boundary condition must be satisfied through the iterative solution.

4 Numerical experiment

As a test problem we consider European two-asset put option with the parameters

$$(\sigma_1, \sigma_2) = (0.1, 0.1), \quad (\mu_1, \mu_2) = (-0.9, -0.9), \quad (\sigma_J^1, \sigma_J^2) = (0.45, 0.45), \quad (\lambda_1, \lambda_2) = (0.1, 0.1),$$
$$(\rho) = 0.3, \quad K = 80, \quad r = 0.05, \quad T = 0.25.$$

In Table 1 we have listed the results of our computations. The method seems to converge, i.e., the error decreases as the number of nodes increases.

Tabel 1: Result for European two-assets option

(S_1, S_2)	MLS (11×11)	MLS (21×21)	MLS (31×31)	MLS (41×41)	FDM
(70,70)	8.1769	8.5909	8.5967	8.6551	8.7328
(80,80)	0.3028	0.8212	1.1044	1.2540	1.2135

In Fig. 1 we have plotted the the meshfree approximation of option price in conjunction with error estimate. In this figure we use finite difference method as benchmark to compute the error estimate of presented MLS algorithm. Finally in Fig. 2, we have plotted the Delta and Gamma Greeks for this option.

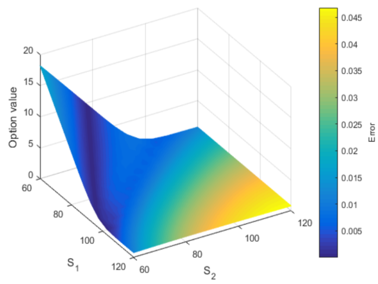

Figure 1: European two-assets basket option values in conjunction with error estimate.

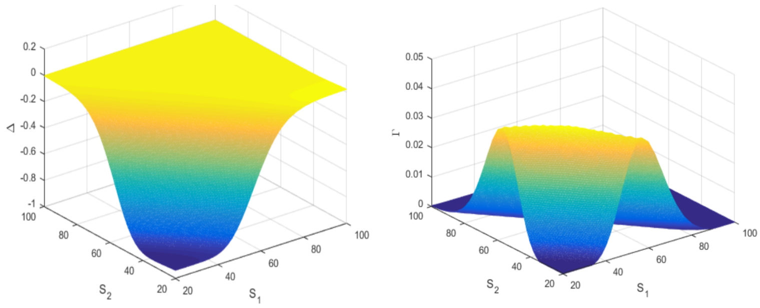

Figure 2: European two-assets Delta and Gamma of basket option.

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