The Fair Valuation of Life Insurance Contracts in a Partially Observed Market

Mohsen Baghi[1]

Department of Mathematics, Yazd University, Yazd, Iran

M. J. Ebadi

Faculty of Marine Science, Chabahar Maritime University, Chabahar, Iran

---

Abstract

Insurance contracts have always been challenging for many years. Default risk is possible for insurance companies and that will affect investor's decisions. The purpose of this article is to price a participating life insurance contract in presence of default risk of the insurance company in a partially observed market. Two cases are considered. At first, we consider the company's default occurs simultaneously within the maturity. In the second case, we suppose the default occurred prior to the maturity. In the next step, we change the partially observed problem to a complete observed one. This is done by filtering and measure change techniques. Finally, we calculate the valuation of insurance contract as a complete observed problem. This pricing is obtained by using the pricing of an up-and-out call option that is one type of barrier options.

Keywords: default, hidden Markov chain, hitting time, risk-neutral measure

AMS Mathematical Subject Classification [2018]: 13D45, 39B42

---

1 Introduction

Our purpose is to price a life insurance contract value in a model with two agents: policyholders and shareholders. In this model, the policyholder and insurance company are investment partners in a way that if the firm reaches a special level, the policyholder gets a percentage from the profit of the company. We consider two cases for our model: the default which occurs in the maturity or before maturity. In the case which the default occurs before the maturity, the time when the company's equity be less than the barrier is the time when we predict the company has defaulted. This dividend is important because we manage the default risk. In the case of the company's default risk is managed, if the company defaults, it can adopt policies for the amount of payments to the policyholders and shareholders. In addition, the value of the insurance contract will be changed after the default.

[1]speaker

2 The model

Suppose $(\Omega, \mathcal{F}, P)$ is a complete probability space, where P is the real-world probability measure. Let Υ denote time index $[0,T]$. Let $\{W_t\}_{t\in\Upsilon}$ denote a standard Brownian motion on the probability space with respect to its natural filtration $\mathcal{F}^W := \{\mathcal{F}_t^W\}_{t\in\Upsilon}$. The state of an economy augmented by a continuous-time hidden Markov chain process $\{X_t\}_{t\in\Upsilon}$ on $(\Omega, \mathcal{F}, P)$ with a finite state space $\chi := (x_1, x_2, \cdots, x_N)$. We suppose that $\{X_t\}_{t\in\Upsilon}$ and $\{W_t\}_{t\in\Upsilon}$ are independent. Since the market is partially observed or incomplete, the economy state is modeled by "hidden" Markov chain.

We consider an insurance company with two types of agents, policyholders and shareholders. The company is without debt and its time horizon is finite (in time T). Suppose that A_0 is the asset initial value and $L_0 = \alpha A_0$ is the initial investment of policyholders (a part of assets of the company provided by the investment of policyholders), $E_0 = (1-\alpha)A_0$ is the initial equity, and A_t is the asset value in time t. We assume that the asset value evolves according to a geometric Brownian motion,

$$dA_t = \mu_t A_t dt + \sigma_t A_t dW_t^P,$$

Where the expected growth rate μ_t and the volatility σ_t of the market value of the insurance company's asset depend on $\{X_t\}_{t\in\Upsilon}$ are described by: $\mu_t =< \mu, X_t >$, $\sigma_t =< \sigma, X_t >$.

In addition, we suppose the riskless interest rate is stochastic and the dynamics of the price process $\{B_t\}_{t\in\Upsilon}$ for the bank account is described by:

$$dB_t = r_t B_t dt, \qquad B_0 = 1.$$

Suppose that Y_t denote the logarithmic return $\ln(\frac{A_t}{A_0})$ from the asset over the time interval $[0,t]$. Write $\{\mathcal{F}_t^X\}_{t\in\Upsilon}$ and $\{\mathcal{F}_t^Y\}_{t\in\Upsilon}$ for the P-augmentation of the natural filtration generated by $\{X_t\}_{t\in\Upsilon}$ and $\{Y_t\}_{t\in\Upsilon}$, respectively. $\{\mathcal{F}_t^Y\}_{t\in\Upsilon}$ is the available information but since $\{X_t\}_{t\in\Upsilon}$ is not directly observable, $\{\mathcal{F}_t^X\}_{t\in\Upsilon}$ is unavailable.

2.1 The company defaults within the maturity

According [1], we consider a multilevel model for the amount of money which the insurance company pays to the policyholders in the case of the default within the maturity. we analyzed the cases which default occurs simultaneously with maturity:

In the case $A_T < L_T^g$, the company is insolvent and is able to utmost pay A_T to the policyholders and nothing to the equity holders. In the cases $L_T^g \leq A_T < \frac{L_T^g}{\alpha_1}$ or $\frac{L_T^g}{\alpha_i} \leq A_T < \frac{L_T^g}{\alpha_{i+1}}$ for $i = 1, \cdots, n$, the company is able to fulfill its commitments and it is possible to pay a guaranteed amount of money to the policyholders and equity holders. This amount of money is depended on the insurance company's asset value. And if $A_T \geq \frac{L_T^g}{\alpha}$, the policyholders receive a bonus for their participation in the company's investments. This bonus known as the participation coefficient is a coefficient of a call option bought by policyholders with maturity price L_T^g.

The policyholders eventually receive $\Theta_L(T)$ in duration of T, assuming no prior bankruptcy concisely

$$\Theta_L(T) = \begin{cases} A_T & A_T < L_T^g \\ L_T^g & L_T^g \le A_T < \frac{L_T^g}{\alpha_1} \\ L_T^g + \delta_1(\alpha_1 A_T - L_T^g) & \frac{L_T^g}{\alpha_1} \le A_T < \frac{L_T^g}{\alpha_2} \\ L_T^g + \delta_1(\alpha_1 A_T - L_T^g) + \delta_2(\alpha_2 A_T - L_T^g) & \frac{L_T^g}{\alpha_2} \le A_T < \frac{L_T^g}{\alpha_3} \\ \vdots & \\ L_T^g + \delta_1(\alpha_1 A_T - L_T^g) + \delta_2(\alpha_2 A_T - L_T^g) + \cdots + \delta_n(\alpha_n A_T - L_T^g) & A_T \ge \frac{L_T^g}{\alpha_n} \end{cases} \tag{1}$$

where $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_n$. briefly:

$$\Theta_L(T) = L_T^g + \delta_1(\alpha_1 A_T - L_T^g)^+ + \delta_2(\alpha_2 A_T - L_T^g)^+ + ... + \delta_n(\alpha_n A_T - L_T^g)^+ - (L_T^g - A_T)^+. \tag{2}$$

2.2 The company defaults prior to the maturity

In the case of default prior to the maturity, we choose a barrier B_t exponentially (because we ensure the default to be occurred in T).
the company pursues its activities until T if

$$\forall t \in [0, T], \qquad A_t > \lambda L_0 e^{r_g t} \triangleq B_t,$$

if not, it is declared bankrupt.
Suppose τ to be the default time, the first time when A_t hits the barrier B_t,

$$\tau := \inf\{t \in [0, T], A_t < B_t\}.$$

Depending on λ, different cases may occur for the company. If $\lambda \geqslant 1$, the firm is able to pay back the money with the guaranteed rate r_g to the policyholders. The residual capital (equal to $(\lambda - 1)L_0 e^{r_g \tau}$ can be used to pay bankruptcy costs or can be distributed between shareholders. If $\lambda < 1$, the firm is totally insolvent and unable to meet its commitments [5].
Generally, the amount of money that policyholders receive in case of early default is:

$$\Theta_L(\tau) = \min(\lambda, 1) L_\tau^g.$$

3 Changing the partially observed problem to a complete observed problem

The valuation of the life insurance contract as a partially observed problem have the below form

$$V(A_t, t) = e^{-r(T-t)} E_t^Q[\Theta_L(T) I_{\tau > T} | \mathcal{F}_t^Y] + E^Q[e^{-r(\tau - t)} \Theta_L(\tau) | \mathcal{F}_t^Y], \tag{3}$$

where E_t^Q denotes expectation with respect to the risk-neutral measure, Q, conditional on the available information until time t.
Using arbitrage theory in continuous time, in an arbitrage-free market, a risk-neutral probability measure exists. The market described by the Markov-modulated GBM model is incomplete in general and, hence, the martingale measure is not unique. According [3], We use a regime switching random Esscher transform to determine an equivalent martingale pricing measure.

We define $\{\mathcal{G}\}_{t\in\Upsilon}$ as the σ-field $\mathcal{F}_t^X \vee \mathcal{F}_t^Y$. Let $\theta_t := \theta(t, X_t)$ denote the regime switching Esscher parameter, which can be written as follows:

$$\theta_t =< \theta, X_t >,$$

where $\theta := (\theta_1, \theta_2, \cdots, \theta_N) \in \mathbb{R}^N$. According [], the regime-switching Esscher transform $\mathbb{Q}_\theta$ measure equivalent to P on $\mathcal{G}_t$ is defined by:

$$\frac{dQ_\theta}{dP}|\mathcal{G}_t = \frac{\exp(\int_0^t \theta_s dY_s)}{\mathbb{E}_P[\exp(\int_0^t \theta_s dY_s)|\mathcal{F}_t^X]}$$

The Radon-Nikodym derivative of the regime-switching Esscher transform is given by

$$\frac{dQ_\theta}{dP}|\mathcal{G}_t = \exp(\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 \sigma_s^2 ds).$$

Let $\{\tilde{\theta}_t\}_{t\geq 0}$ denote a family of risk-neutral regime switching Esscher parameters.According to the asset pricing fundamental theorem, the lack of arbitrage opportunities is substantially equivalent to the existence of an equivalent martingale measure under which the discounted stock price process is a martingale. The market is partially observed. It mean that the economy state $\{X_t\}_{t\in\Upsilon}$ is not adapted with respect to the available information or the observations. Then we must change the problem to a complete observed problem. Here, the martingale condition is given by considering an enlarged filtration as follows:

$$A_0 = \mathbb{E}_{Q_\theta}[\exp(-\int_0^t r_s ds)A_t|\mathcal{F}_t^X] \qquad \text{for any} \quad t \in \Upsilon$$

As in [6], we can determine $\tilde{\theta}_t$ uniquely from the martingale condition:

$$\tilde{\theta}_t = \frac{r_t - \mu_t}{\sigma_t^2} = -\frac{\lambda_t}{\sigma_t},$$

where $\lambda_t := \frac{\mu_t - r_t}{\sigma_t}$ is the market price of risk or unit risk premium of the reference asset at time t.
Let $\{\tilde{\mathcal{G}}\}_{t\in\Upsilon}$ denote the σ-field $\mathcal{F}_T^X \vee \mathcal{F}_t^Y$, for any $t \in \Upsilon$. From the martingale condition, the Radon-Nikodym derivative of the risk-neutral regime switching Esscher measure $Q_{\tilde{\theta}}$ is given by:

$$\frac{dQ_{\tilde{\theta}}}{dP}|\mathcal{G}_t = \exp[\int_0^t (\frac{r_s - \mu_s}{\sigma_s})dW_s - \frac{1}{2}\int_0^t (\frac{r_s - \mu_s}{\sigma_s})^2 ds].$$

By Girsanovs theorem, the process $W_t^{Q_{\tilde{\theta}}} = W_t^P + \int_0^t \frac{r_s - \mu_s}{\sigma_s} ds$ is a standard Brownian motion with respect to $\{\tilde{\mathcal{G}}_t\}_{t\in\tau}$ under $\mathbb{Q}_{\tilde{\theta}}$. Hence, the market values of the insurance company's asset under $\mathbb{Q}_{\tilde{\theta}}$ can be written as

$$dA_t = r_t A_t dt + \sigma_t A_t dW_t^{Q_{\tilde{\theta}}}.$$

It can be seen that a maximization problem including partial information is needed to solve because the regime X_t is not directly observable and the policies can only be based on past information of the insurance company's asset prices. The transformation of it into a control problem of fully observed is our approach. Now, the partially observed problem is reduced to a complete observed problem by filtering.
According [2], We suppose the unit simplex in $\mathbb{R}^N$ is denoted by:

$$\Delta_{N-1} = \{(d^1, d^2, \cdots, d^N) : d^1 + d^2 + \cdots + d^N = 1, d^i \geq 0, i = 1, \cdots, N\}.$$

For a vector $l = (l_1, \cdots, l_N) \in \mathbb{R}^N$, the associated mapping $\hat{l} : \Delta_{N-1} \longrightarrow \mathbb{R}$ is defined as follows:

$$\hat{l}(d) := \sum_{i=1}^N l_i d^i. \tag{4}$$

In addition, the filter probability that the regime X_t is e_i at time t, conditional on the filtration $\mathcal{G}_t^I$, is denoted by

$$p_t^i := \mathbb{P}[X_t = e_i | \mathcal{G}_t^I], \quad i = 1, \cdots, N. \tag{5}$$

We have the useful relationship

$$\mathbb{E}_{\mathbb{P}}[< l, X_t > | \mathcal{G}_t^I] = \hat{p}_t$$

Now by filtering, we obtain the complete observed problem. According the last equations, we have

$$A(t, p_t) = \sum_{i=1}^{N} A(t, e_i) p_t^i = e^{\hat{r}(p_t)t - \frac{1}{2}\hat{\sigma}^2(p_t) + \hat{\sigma}(p_t) W_t^{Q_\theta}} \tag{6}$$

Then the valuation of the life insurance contract is depended on $A(t.P_t)$ and the problem is complete observed. In fact, the problem is equivalent to the valuation of barrier options of type standard knockout call option with a maturity date T, the asset price $A(t, p_t)$ and the exercise price $\frac{L_T^g}{\alpha_i}$, a fixed amount and a European put option. One of arbitrage free price of our life insurance contract at time t was explained in [5].

References

[1] Baghi, M., Delaverkhalafi, A., (2017), *The valuation of life insurance contracts in presence of default risk*, 4th FINACT-IRAN Conference, Tehran.

[2] Capponi, A., Figueroa-Lpez, J., Pascucci, A., (2015)*Dynamic credit investment in partially observed markets*, Finance stochastic, 19, 891-939.

[3] Elliott. R., Chan, L., Kuen Siu. T., (2005), *Option pricing and Esscher transform under regime switching*, Annals of Finance, 1, 423-432.

[4] Grosen, A., and Jrgensen, P. L. (2000). *Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options, and bonus policies,* Insurance: Mathematics and Economics, 26(1), 37-57.

[5] Grosen, A., and Jrgensen, P. L. (2002). *Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework,* Journal of risk and insurance, 69(1), 63-91.

[6] Mamon, R., Elliott, R., (2007) *Hidden Markov Models in Finance*, Springer.

e-mail: `mohsenbaghi@stu.yazd.ac.ir`