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Optimal Control of CVaR Risk Measure in Continuous-Time using Radial Basis Function Collocation Method

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Abstract

In this paper, we consider continuous-time stochastic optimal control problems containing conditional value-at-risk (CVaR) in their objective function. The major difficulty arising from these problems is the time inconsistency. To resolve this issue and based on [C. W. Miller and I. Yang, *Optimal control of Conditional Value-at-Risk in continuous time*, SIAM Journal on Control and Optimization 55.2 (2017): 856-884.], we convert the original formulation into an equivalent bi-level optimization problem. Based on the fact that inner optimization is a standard stochastic control problem, we numerically solve it by a radial basis function collocation method and demonstrate its effectiveness on a concrete application from portfolio optimization under CVaR constraints. We show the convergence of this approximation to the true optimal solution present some numerical experiments concerning computation of the efficient frontier.

Keywords: stochastic optimal control, conditional value-at-risk, Hamilton-Jacobi-Bellman equation, radial basis collocation method

Mathematics Subject Classification [2018]: 13D45, 39B42

1 Introduction

Conditional value-at-risk (CVaR) is a coherent risk measure which has gained much popularity during the last years as a risk management tool in financial and actuarial applications. It measures the conditional expectation of losses above some percentage of the worst-case loss scenarios.

Although CVaR has many useful properties in a risk management context, the main weakness is that CVaR is not a time-consistent risk measure. This feature poses several challenging issues in the context of continuous-time optimization with CVaR constraints.

1.1 Main Problem

The main goal in this paper is to solve the following stochastic optimal control problem with a nonstandard objective:

$$\inf_{A \in \mathcal{A}} \rho(g(X_T^A)), \tag{1}$$

[1]speaker

in which ρ is a fixed extremal risk measure and $g(X_T^A)$ is the state-dependent cost function when the control A is executed. Also $\mathcal{A}$ denotes the set of admissible control strategies and the control $A \in \mathcal{A}$ affects a system state of interest through the stochastic differential equation

$$\begin{aligned} dX_t^A &= \mu(X_t^A, A_t)dt + \sigma(X_t^A, A_t)dW_t, \\ X_0^A &= x_0. \end{aligned}$$

In general, (1) is a time-inconsistent nonlinear stochastic optimal control problem to which we cannot apply dynamic programming. Miller and Yang [3] has shown how to use the structure of extremal risk measures to convert (1) to an equivalent bilevel optimization problem.

Proposition 1.1. *(bilevel optimiaztion). We can write the problem of dynamic optimization over an extremal risk measure as*

$$\inf_{A\in\mathcal{A}} \rho(g(X_T^A)) = \inf_{y\in\mathbb{R}^m} V(y),$$

where V is defined via a standard stochastic optimal control problem of the form

$$V(y) := \inf_{A\in\mathcal{A}} \mathbb{E}[f(g(X_t^A), y)].$$

2 Main results

In this section, we illustrate a practical use of main results and approximation methods in an application to portfolio optimization under mean-CVaR objective. Our goal is to use this methodology to compute the efficient frontier representing the trade-off between maximizing expected log-return and minimizing CVaR of losses.

Consider a market consisting n risky assets, and assume there exists a risk-free asset with drift r. We assume that we choose a control A, representing the percent of the portfolio exposed to each of n risky assets. for simplicity, we consider log value of the portfolio evolves via the SDE

$$dX_t^A = \big[r + A_t^T(\mu - r1) - \frac{1}{2}A_t^T \Sigma A_t\big]dt + A_t^T \Sigma^{1/2} dW_t.$$

Here $\mu \in \mathbb{R}^n$ is a vector of drifts and Σ is the covariance matrix of returns. Then we can interpret X_t^A as the log-return of the portfolio up to time t.

Now, we consider the problem of minimizing a mean-CVaR objevtive:

$$\inf_{A\in\mathcal{A}} [\mathbb{E}[-X_t^A] + \lambda \mathrm{CVaR}_\alpha[-X_t^A]], \tag{2}$$

for fixed $\lambda > 0$ and $\alpha \in (0, 1)$.

2.1 Solution Methodology

From (bi-level optimization) proposition 1.1, the problem (2) could be formulated as a bi-level optimization of the form

$$\inf_{y\in\mathbb{R}} V(y),$$

where

$$V(y) := \inf_{A\in\mathcal{A}} \mathbb{E}[f(g(X_t^A), y)].$$

Here, we take $g(X_T^A) = -X_T^A$ and based on [4] we have

$$f(x, y) := x + \lambda\big(y + \frac{1}{1-\alpha}(x - y)^+\big). \tag{3}$$

Applying the inf-convolution to (3) we obtain

$$f_\epsilon(g(x),y) := \begin{cases} -x + \lambda\big(y - \frac{x+y}{1-\alpha}\big) - \frac{1}{2}\big(\frac{\alpha}{1-\alpha}\lambda\big)^2\epsilon, & x+y < -\frac{\alpha}{1-\alpha}\lambda\epsilon, \\ \frac{1}{2\epsilon}(y+x)^2 - (1+\lambda)x, & -\frac{\alpha}{1-\alpha}\lambda\epsilon \le x+y \le \lambda\epsilon, \\ -x + \lambda y - \frac{1}{2}\lambda^2\epsilon, & x+y > \lambda\epsilon, \end{cases}$$

which is uniformly semi-concave in y.
Also, we consider the perturbed dynamics

$$d\hat{X}_t^{A,\epsilon} = \big[r + A_t^T(\mu - r1) - \frac{1}{2}A_t^T\Sigma A_t\big]dt + A_t^T\Sigma^{1/2}dW_t + \epsilon d\hat{W}_t,$$

and try to solve for the perturbed value function

$$V_\epsilon(y) := \inf_{A\in\hat{\mathcal{A}}} \mathbb{E}[f_\epsilon(g(\hat{X}_T^{A,\epsilon}),y)].$$

2.2 RBF Collocation

We want to solve the HJB equation in n-dimensions of the form

$$\upsilon_t + \inf_{a\in\mathbb{A}} \big[\frac{1}{2}tr(A_t^T\Sigma A_t D_x^2\upsilon) + \big(r + A_t^T(\mu - r1) - \frac{1}{2}A_t^T\Sigma A_t\big)D_x\upsilon\big] = 0 \qquad \text{in} \quad [0,T)\times\mathbb{R}^n,$$
$$\upsilon(T,x) = f(-x,y) \qquad \text{on} \quad \{t=T\}\times\mathbb{R}^n,$$

and then compute the quantity

$$V(y) = \upsilon(0,0).$$

On the other hand, the gradient PDE for $\omega := (\omega^{(1)},\cdots,\omega^{(m)})$ is of the form

$$\omega_t^{(k)} + \frac{1}{2}tr\big(\sigma(x,a^*(t,x))\sigma(x,a^*(t,x))^T D_x^2\omega^{(k)}\big) + \mu(x,a^*(t,x)).D_x\omega^{(k)} = 0 \quad in \quad [0.T)\times\mathbb{R}^n,$$
$$\omega^{(k)}(T,x) = [D_y f(-x,y)]_k \quad on \quad \{t=T\}\times\mathbb{R}^n,$$

for $k = 1,\cdots,m$. that a^* is

$$a^*(t,x) \in \arg\min_{a\in\mathbb{A}} \big[\frac{1}{2}tr(\sigma(x,a)\sigma(x,a)^T D_x^2\upsilon) + \mu(x,a).D_x\upsilon\big], \quad \forall(t,x)\in[0,T)\times\mathbb{R}^n.$$

Furthermore, we have

$$DV(y) = \omega(0,0).$$

Therefore, we can solve the outer optimization problem using a gradient descent method.
For numerical solution we use a radial basis function collocation method based on the inverse multiquadric (IMQ) RBFs defined by

$$\phi_i(x) = \frac{1}{\sqrt{\|x-x_i\|_2^2 + c}}, \quad i = 1,2,\cdots,M,$$

where $\|.\|_2$ is the Euclidean norm, M a positive integer and $c > 0$ is the shape parameter. We defined $\upsilon_r(t,x) = \sum_{i=1}^M \alpha_i(t)\phi_i(x)$. replacing υ in equation with υ_r yields a linear system.

Example 2.1. We consider a concrete example involving selection between a single risky asset and a risk free asset. We choose $\mu = 11\%, \sigma = 20\%, r = 1\%, T = 1, \mathbb{A} := [-6,+6]$ and $\alpha = 95\%$.

Figure 1 illustrate the linear decrease in relative error from approximation as a function of the approximation parameter ϵ. We compute points on the efficient frontier between expected log-return and CVaR by varying λ over the interval (0,1].
In Figure 2, we illustrate a comparison between the efficient frontier under dynamic strategies and under static strategies.

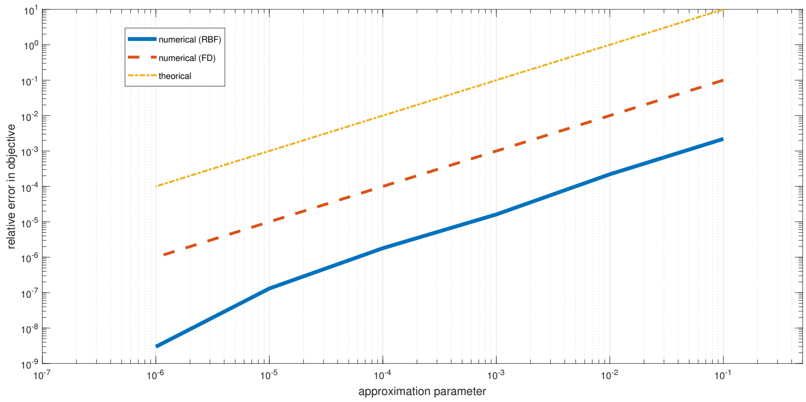

Figure 1: the relative error $|V(y*) - V_\epsilon(y_\epsilon^*)|$ in the objective

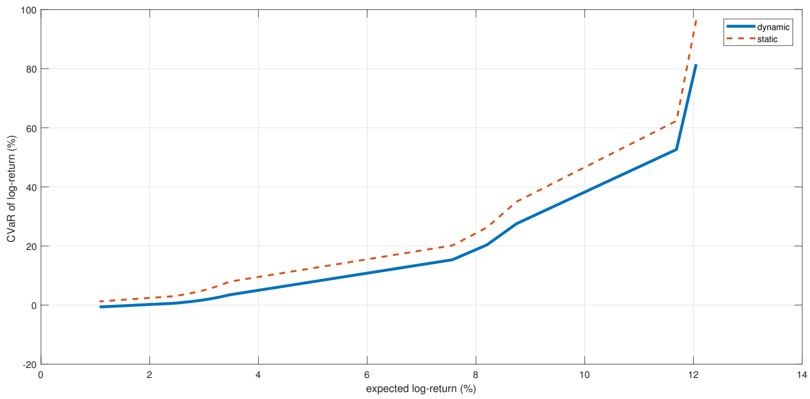

Figure 2: The efficient frontier of mean-CVaR portfolio optimization.

References

[1] Huang, C.S., Wang, S., Chen, C.S. and Li, Z.C., *A radial basis collocation method for Hamilton-Jacobi-Bellman equations*, Automatica 42, 12 (2006), pp. 2201–2207.

[2] Katzourakis, N., *An Introduction to Viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in* L_∞, Springer, 2014.

[3] Miller, C.W. and Yang, I, *Optimal control of conditional value-at-risk in continuous time*, SIAM Journal on Control and Optimization 55, 2 (2017), pp. 856–884.

[4] Rockafellar, R.T. and Uryasev, S., *Optimization of conditional value-at-risk*, Journal of Risk, 2 (2000), pp. 21–42.

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